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Rapid Communication

Large amplitude free vibrations of a mass grounded by linear and nonlinear springs in series

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Abstract

This communication is concerned with the large amplitude free vibration of a mass grounded by linear and nonlinear springs in series. Based on a single equation of motion in terms of relative displacement variable, a qualitative analysis is completed and some new and interesting dynamic behaviors are discovered. These behaviors include oscillations in asymmetric single potential wells and existence of asymptotes in phase plane for the case of softening springs. The ranges of oscillations are determined and expressions of exact periods for symmetric and asymmetric oscillations are established. Furthermore, the construction of analytical approximations to period and periodic solution is briefly described.

1. Introduction

A mechanical system having a mass grounded by two linear springs in series or parallel may be replaced with their equivalents [1,2]. When one of the springs in parallel is linear while the other is nonlinear, it results in an equivalent, nonlinear spring with a larger coefficient for its linear part. On the other hand, if a linear spring is connected with a nonlinear one serially, derivation of an equivalent spring becomes complicated. A single complex nonlinear equation of motion in terms of relative displacement was obtained in Ref. [3], but no qualitative analysis about the equation was presented and no expression of exact period was given. By applying the Lindstedt–Poincaré (LP) method and the classical harmonic balance (HB) method [4,5] to this equation, respectively, Telli and Kopmaz [3] established analytical approximate periodic solution for the case of hardening spring. It has been observed [3] that, for fixed $\varepsilon = 0.5$, numerical and analytical approximate solutions found for both methods are in very good agreement when $v_0 \leqslant 1$ and $0.1 \leqslant \xi \leqslant 10$, where ε is the ratio of the coefficient in nonlinear portion to that in the linear portion in relationship between the deflection of spring and the force acting upon it for the nonlinear spring, v_0 is the initial deflection of nonlinear spring and ξ is the ratio of coefficient of linear portion of the nonlinear spring to that of the linear spring. It has been pointed out that these analytical solutions, especially the LP solutions, have large difference from the numerical solution when $v_0 > 1$ [3]. This can be explained as follows. The LP method applies to weakly

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nonlinear systems only; for their HB solutions with two- and three-term approximations, the coefficients A_3 and A_5 are obtained under the assumptions that $A_3 \ll A_1$ and $A_5 \ll A_1$. In addition, the case of softening springs has not been considered in their paper.

Recently, Lai and Lim [6] applied the linearized harmonic balance (LHB) method [7] to the equation of motion derived in Ref. [3], and obtained three analytical approximations to the periodic solutions. However, it is very difficult to construct higher-order analytical approximations by using the LHB method [7], because it requires analytical solution of complicated nonlinear algebraic equation in terms of unknown frequencies. Though analytical approximate solutions to the symmetric oscillations are constructed in Ref. [6], no restriction is imposed to the oscillation amplitude for the case of softening spring, and especially, no qualitative analysis for the dynamic system was carried out there. In fact, some interesting dynamic behaviors for the case of softening spring have not been discovered. In addition, no expression of exact period for symmetric oscillations is provided in Ref. [6].

In this communication, both cases of hardening and softening springs are considered. Based on a qualitative analysis, some new and interesting dynamic behaviors are opened out. These behaviors include the oscillations in the asymmetric single potential well and existence of asymptotes in phase plane for the case of softening springs. The ranges of oscillations are determined and expressions of exact periods for symmetric and asymmetric oscillations are obtained. Finally, a brief description of the construction of analytical approximate periods and periodic solutions is given.

2. Qualitative analysis

Consider a mechanical system shown in Fig. 1, which has a mass m grounded by linear and nonlinear springs in series [3]. In this figure, the stiffness coefficient of the first linear spring is k_1, the coefficients associated with the linear and nonlinear portions of spring force in the second spring with cubic nonlinear characteristic are described by k_2 and k_3, respectively. Let ε be defined as

$$\varepsilon = k_3/k_2. \tag{1}$$

The case of $k_3 > 0$ corresponds to a hardening spring while $k_3 < 0$ indicates a softening one.

Let x and y denote the absolute displacements of the connection point of two spring, and the mass m, respectively. By introducing two new variables

$$u = y - x, \quad r = x. \tag{2}$$

Telli and Kopmaz [3] obtained the following governing equation for u and r:

$$(1 + 3\varepsilon\eta u^2)u'' + 6\varepsilon\eta u u'^2 + \omega_0^2(u + \varepsilon u^3) = 0, \tag{3}$$

$$r = x = \xi(1 + \varepsilon u^2)u, \quad y = (1 + \xi + \xi\varepsilon u^2)u, \tag{4}$$

where a prime denotes differentiation with respect to time t and

$$\xi = k_2/k_1, \quad \eta = \frac{\xi}{1+\xi}, \quad \omega_0^2 = \frac{k_2}{m(1+\xi)}.$$

Eq. (3) is an ordinary differential equation in u. In the present study, it is assumed that $x'(0) = y'(0) = 0$ then $u'(0) = 0$. For Eq. (3), we consider the following initial conditions:

$$u(0) = A, \quad u'(0) = 0. \tag{5}$$

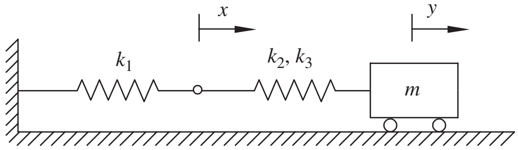

Fig. 1. System with linear and nonlinear springs in series.

Apparently, once u in Eqs. (3) and (5) is solved, variables x and y can be achieved by using Eq. (4). Hence, we will study Eqs. (3) and (5) only.

Premultiplying Eq. (3) with $(1+3\varepsilon\eta u^2)$, integrating the resulting equation, and using initial condition in Eq. (5) yields the following equation:

$$(1+3\varepsilon\eta u^2)^2 u'^2 + V(u) = V(A), \tag{6a}$$

where

$$V(u) = \omega_0^2[u^2 + \varepsilon u^4(1+3\eta)/2 + \varepsilon^2\eta u^6]. \tag{6b}$$

The function $V(u)$ has singular points at u_i, where

$$\frac{\mathrm{d}V(u_i)}{\mathrm{d}u} = 2\omega_0^2\left[u_i + \varepsilon(1+3\eta)u_i^3 + 3\varepsilon^2\eta u_i^5\right] = 0$$

or

$$u_1 = 0 \quad \text{if } \varepsilon > 0,$$

$$u_1 = 0, \quad u_{2,3} = \pm\sqrt{-1/\varepsilon}, \quad u_{4,5} = \pm\sqrt{-1/(3\varepsilon\eta)} \text{ if } \varepsilon < 0. \tag{6c}$$

Note that the singular points u_4, u_5 of $V(u)$ are not "truly" singular points of Eq. (3) since $u = u_{4,5}$ do not satisfy this equation, and in fact, they are added roots resulted from integrating Eq. (3).

The nature of the singular points can be determined by examining $\mathrm{d}^2V(u_i)/\mathrm{d}u^2$. It follows from Eqs. (6a–c) that there are four cases to be considered: $\varepsilon > 0$; $\varepsilon < 0$, $\eta < 1/3$; $\varepsilon < 0$, $\eta = 1/3$ and $\varepsilon < 0$, $\eta > 1/3$. These are discussed individually.

When the nonlinear spring is a hardening one i.e.,$\varepsilon > 0$, $V(u)$ has one singular point only, a minimum at $u = 0$. Thus $u = 0$ is a center. The possible motions in the neighborhood of this point are represented in Fig. 2.

When $\varepsilon < 0$ and $\eta < 1/3$, $V(u)$ arrives its minimum value at u_1, and maximum values at u_2 and u_3; hence u_1 is a center, u_2, u_3 are saddle points. The possible motions are represented in Fig. 3. Note that $u = \pm\sqrt{-1/(3\varepsilon\eta)}$ are two asymptotes.

As η increases, two saddle points and two asymptotes approach each other, respectively. They coalesce at $u = \pm\sqrt{-1/\varepsilon}$ when $\eta = 1/3$. Hence, for the case of $\eta = 1/3$, this system has a minimum at $u = 0$ only, and it is a center. The possible motions are represented in Fig. 4.

Finally when $\varepsilon < 0$ and $\eta > 1/3$, $V(u)$ arrives its minimum values at u_1, u_2 and u_3, respectively; hence u_1, u_2 and u_3 are centers. The possible motions are represented in Fig. 5.

Based on the qualitative analysis above for Eq. (3), the periodic motion of this system depends upon the initial oscillation amplitude A and values of the parameters ε and η. In the case of $\varepsilon > 0$, the system will oscillate

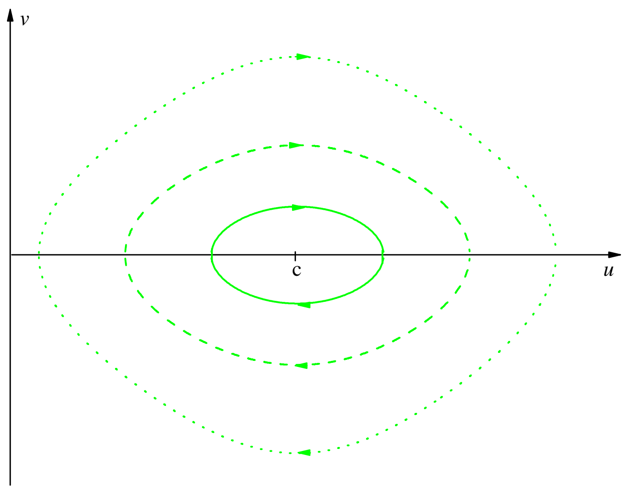

Fig. 2. Phase plane for $\varepsilon > 0$.

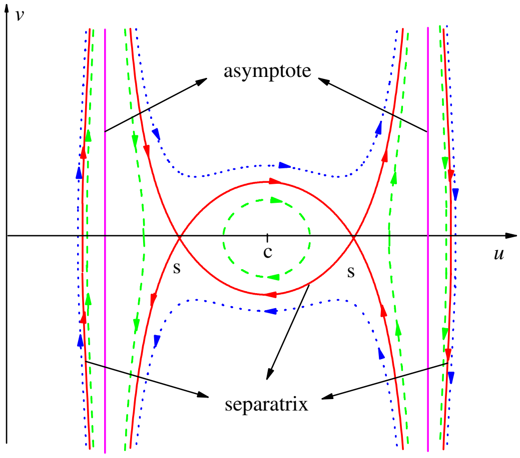

Fig. 3. Phase plane for $\varepsilon < 0, \eta < 1/3$.

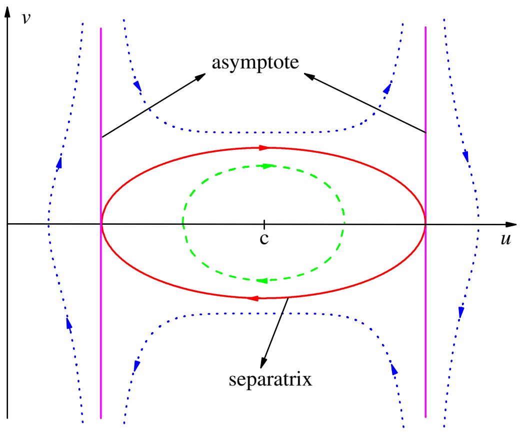

Fig. 4. Phase plane for $\varepsilon < 0, \eta = 1/3$.

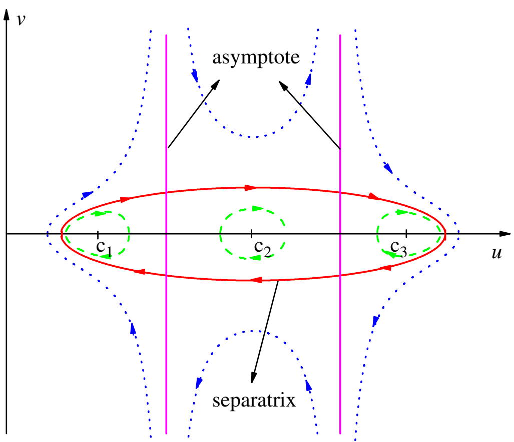

Fig. 5. Phase plane for $\varepsilon < 0, \eta > 1/3$.

between the symmetric bounds $[-A,A]$ and $0<A<+\infty$. For the case of $\varepsilon<0$ and $\eta\leqslant 1/3$, oscillation occurs around stable equilibrium point $u=0$ only, which is symmetric about this point and oscillation amplitude should satisfy $A<\sqrt{-1/\varepsilon}$. For the case of $\varepsilon<0$ and $\eta>1/3$, oscillation may occur around stable equilibrium point $u=0$ and is symmetric about this point, and oscillation amplitude A is subjected to $A<\sqrt{-1/(3\varepsilon\eta)}$; oscillation may also occur around equilibrium point $u=\sqrt{-1/\varepsilon}(u=-\sqrt{-1/\varepsilon})$ and is asymmetric about it, and left (right) oscillation amplitude should satisfy $\sqrt{-1/(3\varepsilon\eta)}<A<\sqrt{-1/\varepsilon}(-\sqrt{-1/\varepsilon}<A<-\sqrt{-1/(3\varepsilon\eta)})$. The ranges of oscillations are then determined for various cases.

3. Expression of exact periods and construction of analytical approximate solutions

3.1. Oscillations for $\varepsilon>0$ or $\varepsilon<0, \eta\leqslant 1/3, A<\sqrt{-1/\varepsilon}$ or $\varepsilon<0$, $\eta>1/3$, $A<\sqrt{-1/(3\varepsilon\eta)}$

Oscillations for the case of $\varepsilon>0$ or $\varepsilon<0$, $\eta\leqslant 1/3$, $A<\sqrt{-1/\varepsilon}$ or $\varepsilon<0$, $\eta>1/3$, $A<\sqrt{-1/(3\varepsilon\eta)}$ are first considered, which occur between symmetric limits $[-A,A]$. Based on Eqs. (6a, b), the exact period for such an oscillation can be expressed as follows:

$$T_e(A)=4\int_0^{\pi/2}\frac{1+3\varepsilon\eta A^2\sin^2\varphi}{\omega_0\sqrt{1+\varepsilon A^2(1+3\eta)(1+\sin^2\varphi)/2+\varepsilon^2\eta A^4(1+\sin^2\varphi+\sin^4\varphi)}}\,\mathrm{d}\varphi. \tag{7}$$

The exact period is an implicit function of oscillation amplitude and related parameters. Similarly, the corresponding periodic solution is also an implicit function. Such an implicit solution is not convenient for use. While analytical approximations can supply explicit expressions of the solution and allow the direct discussion of the influence of oscillation amplitude and related parameters on the solution.

Both of LHB method [7] and NHB method [8] can be used to establish the analytical approximate periods and corresponding periodic solutions. The accurate results obtained by LHB [7] have been reported for the case above by Lai and Lim [6], so results of the Newton-harmonic balance method [8] are not shown here. Note that the LHB method [7] results in a complex nonlinear algebraic equation in terms of unknown frequency and its analytical solution is difficult. The NHB method [8] is established by successfully linearizing the governing equation and, subsequently, appropriately imposing the HB method in order to obtain linear algebraic equations instead of nonlinear algebraic equations. In this communication, the construction of analytical approximate solutions will be based on the NHB method [8].

3.2. Oscillations for $\varepsilon<0$, $\eta>1/3$, $\sqrt{-1/\varepsilon}<A<\sqrt{(1-9\eta)/6\varepsilon\eta}$

In the case of $\varepsilon<0$, $\eta>1/3$, $\sqrt{-1/\varepsilon}<A<\sqrt{(1-9\eta)/(6\varepsilon\eta)}$, the oscillation occurs around stable equilibrium points $u=\sqrt{-1/\varepsilon}$, and is asymmetric about it. Hence, we introduce a new variable

$$w=u-\sqrt{-1/\varepsilon}. \tag{8}$$

Substituting Eq. (8) into Eqs. (3) and (5) yields

$$P_1(w)w''+P_2(w)w'^2+P_3(w)=0,\quad w(0)=\hat{A},\quad w'(0)=0, \tag{9a}$$

where

$$P_1(w)=1-3\eta-6\sqrt{-\varepsilon}\eta w+3\varepsilon\eta w^2,\quad P_2(w)=6\varepsilon\eta w-6\sqrt{-\varepsilon}\eta,$$
$$P_3(w)=\omega_0^2\left(\varepsilon w^3-2w-3\sqrt{-\varepsilon}w^2\right),\quad \hat{A}=A-\sqrt{-1/\varepsilon}. \tag{9b}$$

The system in Eqs. (9a, b) will oscillate between the asymmetric bounds $[-\hat{B},\hat{A}]$, and $\hat{B}(\hat{B}>0)$ is equal to $\hat{B}=\sqrt{-1/\varepsilon}-B(B>0)$ and B satisfies

$$V(A)=V(B)\quad\left(\sqrt{-1/(3\varepsilon\eta)}<B<\sqrt{-1/\varepsilon}\right), \tag{10}$$

where $V(u)$ is given in Eq. (6b). Using Eq. (10), we can obtain $\hat{B}(\hat{B}>0)$ in terms of $\hat{A}$ as follows:

$$\hat{B} = \sqrt{\frac{-1}{\varepsilon}} + \frac{1}{2\varepsilon\sqrt{\eta}}\sqrt{-\varepsilon(1+3\eta) - 2\varepsilon^2\eta\left(\sqrt{-\frac{1}{\varepsilon}} + \hat{A}\right)^2 - \varepsilon\sqrt{\Delta}}, \tag{11}$$

where

$$\Delta = (1 - 10\eta + 9\eta^2) - 4\varepsilon\eta\left(\sqrt{-\frac{1}{\varepsilon}} + \hat{A}\right)^2\left[1 + 3\eta + 3\varepsilon\eta\left(\sqrt{-\frac{1}{\varepsilon}} + \hat{A}\right)^2\right].$$

In a way similar to Refs. [9–11], we introduce the two new nonlinear oscillating systems which oscillate between the symmetric bounds $[-H, H]$

$$\Psi_1(w,\alpha)w'' + \Psi_2(w,\alpha)w'^2 + \Psi_3(w,\alpha) = 0, \quad w(0) = H, \quad w'(0) = 0, \tag{12a}$$

where

$$\Psi_1(w,\alpha) = \begin{cases} 1 - 3\eta - 6\alpha\sqrt{-\varepsilon}\eta w + 3\varepsilon\eta w^2 & \text{if } w \geqslant 0, \\ 1 - 3\eta + 6\alpha\sqrt{-\varepsilon}\eta w + 3\varepsilon\eta w^2 & \text{if } w < 0, \end{cases}$$

$$\Psi_2(w,\alpha) = \begin{cases} 6\varepsilon\eta w - 6\alpha\sqrt{-\varepsilon}\eta & \text{if } w \geqslant 0, \\ 6\varepsilon\eta w + 6\alpha\sqrt{-\varepsilon}\eta & \text{if } w < 0, \end{cases}$$

$$\Psi_3(w,\alpha) = \begin{cases} \omega_0^2(\varepsilon w^3 - 2w - 3\alpha\sqrt{-\varepsilon}w^2) & \text{if } w \geqslant 0, \\ \omega_0^2(\varepsilon w^3 - 2w + 3\alpha\sqrt{-\varepsilon}w^2) & \text{if } w < 0 \end{cases} \tag{12b}$$

and $\alpha = \pm 1$. Here we set $H = \hat{A}$ for $\alpha = 1$, and $H = \hat{B}$ for $\alpha = -1$, respectively.

Following the method adopted in Refs. [10,11], we may construct the corresponding analytical approximate periods and periodic solutions to Eqs. (9a, b). To save space, the corresponding analytical approximation is omitted.

By integrating Eqs. (12a, b), we get the exact period $T_e(\hat{A})$ as follows:

$$T_e(\hat{A}) = \int_0^{\pi/2} \frac{2F_1(\hat{A})}{\omega_0\sqrt{F_2(\hat{A})}}\,\mathrm{d}t + \int_0^{\pi/2} \frac{2F_1(-\hat{B})}{\omega_0\sqrt{F_2(-\hat{B})}}\,\mathrm{d}t, \tag{13}$$

where

$$F_1(z) \equiv -1 + 3\eta + 6\sqrt{-\varepsilon}\eta z \sin t - 3\varepsilon\eta z^2 \sin^2 t,$$

$$F_2(z) \equiv (6\eta - 2) + 2z\sqrt{-\varepsilon}(7\eta - 1)\left(\frac{1 + \sin t + \sin^2 t}{1 + \sin t}\right) + \frac{1}{2}\varepsilon z^2(1 - 27\eta)(1 + \sin^2 t)$$
$$- 6\varepsilon\sqrt{-\varepsilon}z^3\left(\frac{1 + \sin t + \sin^2 t + \sin^3 t + \sin^4 t}{1 + \sin t}\right) + \varepsilon^2\eta z^4(1 + \sin^2 t + \sin^4 t).$$

Since the periodic motion around the equilibrium point $u = -\sqrt{-1/\varepsilon}$ is similar to that around the equilibrium point $u = +\sqrt{-1/\varepsilon}$. The results above may easily be transformed the one for oscillation amplitude $-\sqrt{((1-9\eta)/6\varepsilon\eta)} < A < -\sqrt{-1/\varepsilon}$.

4. Conclusions

Some new and interesting dynamic behaviors for large amplitude free vibrations of a system that consists of a mass grounded linear and nonlinear springs in series have been discovered. The analysis is based on a single equation of motion in terms of relative displacement variable. Both cases of hardening and softening cubic nonlinear spring have been dealt with. The ranges of oscillations have been determined and expressions of exact periods for symmetric and asymmetric oscillations have been established. A brief description for constructing analytical approximate solutions has been given.

Acknowledgments

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